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L-functions of holomorphic cusp forms on U(2, 1)

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0. Introduction

In this note, we report several results on the standard L-functions of holomorphic cusp forms on the unitary group H of hermitian forms of signature (2, 1). The L-functions we investigate here are associated to the 6-dimensional representation of the L-group ${}^{L}H$ of H that is induced from the standard representation of ${}^{L}H^{o} = GL_3(\mathbf{C})$. Such L-functions has been studied by several mathematicians; for example, see Shintani [9], Gelbart and Piatetski-Shapiro[3], Kudla [5], Gelbart and Rogawski [4].

In §1 and §2, we recall basic facts about holomorphic cusp forms on the unitary groups. In §3, we recall the definition of the Hecke algebra of H and introduce the local Euler factor at each rational prime. After defining the global L-function $L(F; s)$ for a Hecke eigenform F and its gamma factor in §4, we state one of the main results of the paper in §5: the holomorphy and functional equation of $L(F; s)$ (here we have to impose a certain technical assumption on F). In the final section, we give a partial result on the critical values of $L(F; s)$. The method of proof is based on a certain integral expression of the L-functions studied in our previous paper [6]. Details will appear elsewhere.

1. Unitary groups

Let $K = \mathbf{Q}(\sqrt{d_K})$ be an imaginary quadratic field of discriminant $d_K < 0$ and $\mathcal{O}_K$ the ring of integers in K. Denote by σ the non-trivial automorphism of K.

Let $\{1, \theta\}$ be a $\mathbf{Z}$-basis of O_K with $\operatorname{Im}\theta > 0$ and put $\kappa = \theta - \theta^\sigma\ (= \sqrt{d_K}\,)$. Let $H = U(T)$ be the unitary group of a skew hermitian matrix $T = \begin{bmatrix} & & -1 \\ & -\kappa & \\ 1 & & \end{bmatrix}$: $H_\mathbf{Q} = \{ h \in GL_3(K) \mid {}^t h^\sigma\, T\, h = T \}$. Note that the signature of a hermitian matrix κT is $(1, 2)$.

Let $N_\mathbf{Q} = \{n(w, v) = \begin{bmatrix} 1 & \kappa w^\sigma & v + \frac{\kappa}{2} w w^\sigma \\ & 1 & w \\ & & 1 \end{bmatrix} \mid w \in K, v \in \mathbf{Q}\}$ and $M_\mathbf{Q} = \{\begin{bmatrix} t & & \\ & \mu & \\ & & (t^\sigma)^{-1} \end{bmatrix} \mid t \in K^\times, \mu \in K^1\}$, where $K^1 = \{\mu \in K^\times \mid \mu\mu^\sigma = 1\}$. Then $P = NM$ is a maximal parabolic subgroup of H.

Let $D = \{ Z = \begin{bmatrix} z \\ w \end{bmatrix} \in \mathbf{C}^2 \mid \frac{1}{\kappa}(z - \bar{z}) - |w|^2 > 0 \}$ be a symmetric domain and put $Z^\sim = \begin{bmatrix} z \\ w \\ 1 \end{bmatrix} \in \mathbf{C}^3$ for $Z = \begin{bmatrix} z \\ w \end{bmatrix} \in D$. Then the action of $H_\infty = H(\mathbf{R})$ on D: $(h, Z) \mapsto h\langle Z \rangle$ is given by $h \cdot Z^\sim = (h\langle Z\rangle)^\sim \cdot J_H(h, Z)$, where $J_H(h, Z) \in \mathbf{C}^\times$. Denote by U_∞ the isotropy subgroup of $Z_\theta = \begin{bmatrix} \theta \\ 0 \end{bmatrix} \in D$ in H_∞.

For a rational prime p, we write $K_p = K \otimes_\mathbf{Q} \mathbf{Q}_p$ and $O_{K,p} = O_K \otimes_\mathbf{Z} \mathbf{Z}_p$. Put $H_p = H(\mathbf{Q}_p)$, $U_p = H_p \cap GL_3(O_{K,p})$ and $U_p^* = \{ h \in U_p \mid (h-1)T^{-1} \in M_3(O_{K,p}) \}$. Then U_p^* is a normal subgroup of U_p and $[U_p : U_p^*]$ is equal to 1 if $p \nmid d_K$ and 2 if $p \mid d_K$. Note that the Iwasawa decomposition $H_p = N_p M_p U_p^*$ holds. We normalize the Haar measure dh on H_p by

$$\int_{H_p} f(h)\, dh = \int_{N_p} dn \int_{M_p} dm \int_{U_p^*} du^*\, f(nmu^*)\, \delta(m)^{-1},$$

where f is any integrable function on H_p, $\delta(m) = d(mnm^{-1})/dn$ and the Haar measures dn, dm and du^* on N_p, M_p and U_p^* are normalized so that $\mathrm{vol}(N_p \cap U_p^*) = \mathrm{vol}(M_p \cap U_p^*) = \mathrm{vol}(U_p^*) = 1$. We normalize the Haar measure dh_∞ on H_∞ by

$$\int_{H_\infty} f(h_\infty)\, dh_\infty = \int_D f^\sim(Z)\, d\mu(Z),$$

where $f^{\sim}(h_\infty\langle Z_\theta\rangle) = \int_{U_\infty} f(h_\infty u_\infty)\, du_\infty$ for $h_\infty \in H_\infty$ and $d\mu(Z) = (\frac{1}{\kappa}(z - \overline{z}) - |w|^2)^{-3}\, d\mathrm{Re}(z)\, d\mathrm{Im}(z)\, d\mathrm{Re}(w)\, d\mathrm{Im}(w)$. The Haar measure dh on $H_{\mathbf{A}}$ is defined to be the product measures $\prod_{v\leq\infty} dh_v$. We set $U^*_{\mathbf{A}_f} = \prod_{p<\infty} U^*_p$ and $U^*_{\mathbf{A}} = U_\infty \cdot U^*_{\mathbf{A}_f}$.

2. Automorphic forms

Let ℓ be a positive integer with $\ell \equiv 0 \pmod{|O_K^\times|}$. Let $S_\ell(U^*_{\mathbf{A}})$ be the space of holomorphic cusp forms on H of weight ℓ defined as follows:

$S_\ell(U^*_{\mathbf{A}}) = \{ F : H_{\mathbf{Q}}\backslash H_{\mathbf{A}}/U^*_{\mathbf{A}_f} \to \mathbf{C} \mid$

(i) $F(hu_\infty) = F(h)\cdot J_H(u_\infty, Z_\theta)^{-\ell}$ for $u_\infty \in U_\infty$.

(ii) The function $h_\infty \to F(h_\infty h_f)\cdot J_H(h_\infty, Z_\theta)^{\ell}$ gives rise to a holomorphic function of $h_\infty\langle Z_\theta\rangle \in D$ for any $h_f \in H_f$.

(iii) F is bounded on $H_{\mathbf{A}}$ $\}$.

It is known that $F \in S_\ell(U^*_{\mathbf{A}})$ satisfies the cuspidal condition

$$\int_{N_{\mathbf{Q}}\backslash N_{\mathbf{A}}} F(nh)\, dn = 0$$

for any $h \in H_{\mathbf{A}}$. The Petersson inner product of $S_\ell(U^*_{\mathbf{A}})$ is defined by

$$\langle F, F'\rangle = \int_{H_{\mathbf{Q}}\backslash H_{\mathbf{A}}} F(h)\,\overline{F'(h)}\, dh \qquad F, F' \in S_\ell(U^*_{\mathbf{A}}).$$

3. Hecke algebra

For a rational prime p, let $\mathcal{H}_p$ be the algebra of compactly supported bi U^*_p-invariant functions on H_p. The object of this section is to recall Satake's parametrization of $\mathrm{Hom}_{\mathbf{C}}(\mathcal{H}_p, \mathbf{C})$ (cf. [7]).

First we consider the case where $\left(\frac{K/\mathbf{Q}}{p}\right) \neq 1$ and hence K_p is a field. Put $K^1_p = \{ \mu \in K^\times_p \mid \mu\,\mu^\sigma = 1\}$ and $K^1_p(\kappa) = \{ \mu \in K^1_p \mid \frac{\mu-1}{\kappa} \in O_{K,p} \}$. It is easy to see that

U_p/U_p^* is isomorphic to $K_p^1 / K_p^1(\kappa)$, which is trivial if $\left(\frac{K/Q}{p}\right) = -1$ and the cyclic group of order 2 if $\left(\frac{K/Q}{p}\right) = 0$ (in this case, $K_p^1 / K_p^1(\kappa)$ consists of 1 and π/π^σ where π is a prime element of K_p). Let χ_1 be an unramified character of $K_p^\times$ and χ_o a character of $K_p^1 / K_p^1(\kappa)$. For a pair $\chi = (\chi_o, \chi_1)$, we define a function ϕ_χ on H_p by

$$\phi_\chi(n\begin{bmatrix} t & & \\ & \mu & \\ & & (t^\sigma)^{-1} \end{bmatrix}u^*) = \chi_o(\mu)\,\chi_1(t)\;|t\,t^\sigma|_p$$

for $n \in N_p$, $t \in K_p^\times$, $\mu \in K_p^1$ and $u^* \in U_p^*$. Here $|\cdot|_p$ denotes the normalized valuation of $Q_p^\times$.

We next consider the case $\left(\frac{K/Q}{p}\right) = 1$. Once and for all we fix an isomorphism of K_p onto $Q_p \oplus Q_p$. Then $H_p = \{\, h = (h_1, h_2) \in GL_3(Q_p) \times GL_3(Q_p) \mid {}^th_2\, T_1\, h_1 = T_1 \,\}$, where T_1 is the first component of $T \in GL_3(K_p) = GL_3(Q_p) \times GL_3(Q_p)$. In what follows we identify H_p with $GL_3(Q_p)$ via $h \to h_1$ and identify $\mathcal{H}_p$ with $\mathcal{H}_p(GL_3(Q_p), GL_3(Z_p))$. For a triplet $\chi = (\chi_1, \chi_2, \chi_3)$ of unramified characters of $Q_p^\times$, put

$$\phi_\chi(\begin{bmatrix} t_1 & * & * \\ & t_2 & * \\ & & t_3 \end{bmatrix}u) = \prod_{j=1}^{3} |t_j|_p^{2-j}\,\chi_j(t_j)$$

for $t_j \in Q_p^\times$ $(1 \le j \le 3)$ and $u \in GL_3(Z_p)$.

In both cases, we put

$$\chi^\wedge(\varphi) = \int_{H_p} \phi_\chi(h)\,\varphi(h^{-1})\,dh \qquad\qquad \varphi \in \mathcal{H}_p.$$

Then $\varphi \to \chi^\wedge(\varphi)$ defines an algebra homomorphism of $\mathcal{H}_p$ to $\mathbf{C}$. Moreover every algebra homomorphism of $\mathcal{H}_p$ to $\mathbf{C}$ is of the form $\chi^\wedge$ for some χ.

Let $\Lambda \in \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}_p, \mathbf{C})$ and ω be an unramified character of $K_p^\times$. Choose χ so that $\Lambda = \chi^\wedge$. Denote by ω' the first component of ω under the identification

$K_p^\times = Q_p^\times \times Q_p^\times$ if $\left(\frac{K/Q}{p}\right) = 1$. We define a local L-factor $L_p(\Lambda\otimes\omega; s)$ attached to Λ and ω to be

$$L_p(\Lambda\otimes\omega; s)^{-1}$$

$$= \begin{cases} (1-(\chi_1\omega')(p)p^{-s})\,(1-(\chi_1\omega')^{-1}(p)p^{-s})\,(1-(\chi_2\omega')(p)p^{-s}) & \\ \times\,(1-(\chi_2\omega')^{-1}(p)p^{-s})\,(1-(\chi_3\omega')(p)p^{-s})\,(1-(\chi_3\omega')^{-1}(p)p^{-s}) & \text{if } \left(\frac{K/Q}{p}\right) = 1 \\ (1-(\chi_1\omega)(p)p^{-2s})\,(1-(\chi_1\omega)^{-1}(p)p^{-2s})\,(1-\omega(p)p^{-2s}) & \text{if } \left(\frac{K/Q}{p}\right) = -1 \\ (1-(\chi_1\omega)(\pi)p^{-s})\,(1-(\chi_1\omega)^{-1}(\pi)p^{-s})\,(1-\omega(\pi)\chi_o(\pi/\pi^\sigma)p^{-s}) & \text{if } \left(\frac{K/Q}{p}\right) = 0\,. \end{cases}$$

4. Automorphic L-functions

Fix a Hecke character ω of $K^\times\backslash K_A^\times$ that is unramified everywhere (namely ω is trivial on $\prod_{p<\infty} O_{K,p}^\times$) and satisfies $\omega(x_\infty) = \left(\frac{x_\infty}{|x_\infty|}\right)^\ell$ for $x_\infty \in K_\infty^\times = C^\times$. Let $F \in S_\ell(U_A^*)$ be a Hecke eigenform corresponding to $\Lambda_p \in \mathrm{Hom}_C(H_p, C)$ for each p. That is to say, we have $(F*\varphi_p)(h) := \int_{H_p} F(hx)\,\varphi_p(x^{-1})\,dx = \Lambda_p(\varphi_p)\,F$ for every p and every $\varphi_p \in H_p$ (cf. **3**). The global L-function attached to F and ω is defined by $L(F\otimes\omega; s) = \prod_{p<\infty} L_p(\Lambda_p\otimes\omega_p; s)$, where ω_p is the local component of ω at p. The gamma factor for $L(F\otimes\omega; s)$ is given by

$$L_\infty(F\otimes\omega; s) = (2\pi)^{-3s}\,|d_K|^{\frac{3}{2}s}\,\Gamma(s+\tfrac{\ell}{2})\,\Gamma(s+\tfrac{\ell}{2}-1)^2.$$

We put $\xi(F\otimes\omega; s) = L_\infty(F\otimes\omega; s)\,L(F\otimes\omega; s)$.

5. Functional equation

Let $G = U(S)$ be the unitary group of $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and embed G into H via $\iota_o(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} a & & b \\ & 1 & \\ c & & d \end{bmatrix}$. Then, for $F \in S_\ell(U_A^*)$, the pullback $\iota_o^* F(g) = F(\iota_o(g))$ by ι_o is an automorphic form on G. Our first main result is as follows:

Theorem 1 *Let* $F \in S_\ell(U_A^*)$ *be a Hecke eigenform. Assume that* $\ell > 4$ *and that* $\iota_o^* F$ *is not identically equal to zero. Then* $\xi(F \otimes \omega; s)$ *can be continued to an entire function of* s *on* $\mathbf{C}$ *and satisfies the functional equation*

$$\xi(F \otimes \omega; s) = \xi(F \otimes \omega; 1-s).$$

6. Special values of automorphic L-functions

In this section, we assume that the class number of K is one. Note that the Hecke character ω is uniquely determined in this case. Let $\overline{\mathbf{Q}}$ be the algebraic closure of K in $\mathbf{C}$. For $F \in S_\ell(U_A^*)$, we put $F^{dm}(Z) = F(h_Z)\, J_H(h_Z, Z_\theta)^\ell$ where h_Z is any element of H_∞ such that $h_Z\langle Z_\theta\rangle = Z \in D$. Then F^{dm} is a holomorphic function on D that satisfies $F^{dm}(\gamma\langle Z\rangle) = J_H(\gamma, Z)^\ell\, F^{dm}(Z)$ for $\gamma \in \Gamma^* = \{\gamma \in H_Z \mid (\gamma - 1)T^{-1} \in M_3(O_K)\}$. It follows that F^{dm} admits the Fourier-Jacobi expansion $F^{dm}(\begin{bmatrix} z \\ w \end{bmatrix}) = \sum_{r=1}^{\infty} g_r(w)\, e[rz]$, where we put $e[x] = \exp(2\pi i x)$ for $x \in \mathbf{C}$. We say that F is $\overline{\mathbf{Q}}$*-rational* if $g_r(v)\, e[\frac{r\kappa}{2} v v^\sigma] \in \overline{\mathbf{Q}}$ for any $v \in K$ and any $r \geq 1$. By virtue of Shimura ([8]), the space $S_\ell(U_A^*)_{\overline{\mathbf{Q}}}$ of $\overline{\mathbf{Q}}$-rational forms of weight ℓ on U_A^* spans $S_\ell(U_A^*)$. Let $L(\omega; s)$ be the Hecke L-function of K attached to ω. The second main result of this note is stated as follows:

Theorem 2 *Assume that the class number of* K *is one and that* $\ell > 4$. *Let* $F \in S_\ell(U_A^*)$ *be a* $\overline{\mathbf{Q}}$*-rational Hecke eigenform with* $\iota_o^* F \not\equiv 0$. *Then there exists a* $\overline{\mathbf{Q}}$*-rational Hecke eigenform* $F' \in S_\ell(U_A^*)$ *with the same eigenvalues as* F *such that*

$$\xi(F\otimes\omega; \frac{\ell}{2} - 1) = c\cdot\pi^{\frac{3}{2}\ell} L(\omega; \frac{\ell}{2}) <F', F'>$$

with a non-zero constant $c \in \overline{\mathbf{Q}}^{\times}$.

Remark. The set of the critical points (in the sense of [1]) of $\xi(F\otimes\omega; s)$ is $\{ k \mid 2 - \frac{\ell}{2} \leq k \leq \frac{\ell}{2} - 1 \}$.

In view of Garrett's results on Petersson inner products of arithmetic Siegel modular forms ([2]), the following conjecture seems to be plausible.

Conjecture *Let* $F, F' \in S_{\ell}(U_{\mathbf{A}}^{*})$ *be* $\overline{\mathbf{Q}}$*-rational Hecke eigenforms with the same Hecke eigenvalues. Then we have* $\dfrac{<F, F>}{<F', F'>} \in \overline{\mathbf{Q}}$.

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